Chapter 14

Graphs

14.1 Introduction

A *graph* G is an ordered pair (V, E), where V is a set and E is a set of two-element subsets of V. That is,

$$E \subseteq \big\{\{x, y\} : x, y \in V, x \neq y\big\}.$$

Elements of V are the *vertices* (sometimes called *nodes*) of the graph and elements of E are the *edges*. If $e = \{x, y\} \in E$ we say that x and y are *adjacent* in the graph G, that y is a *neighbor* of x in G and vice versa, and that the edge e is *incident* to x and y.

What are graphs good for? Graphs are perhaps the most pervasive abstraction in computer science. It is hard to appreciate their tremendous usefulness at first, because the concept itself is so elementary. This appreciation comes through uncovering the deep and fascinating theory of graphs and its applications.

Graphs are used to model and study transportation networks, such as the network of highways, the London Underground, the worldwide airline network, or the European railway network; the 'connectivity' properties of such networks are of great interest. Graphs can also be used to model the World Wide Web, with edges corresponding to hyperlinks; Google uses sophisticated ideas from graph theory to assign a PageRank to every vertex of this graph as a function of the graph's global properties. In this course we will introduce the basic concepts and results in graph theory, which will allow you to study and understand more advanced techniques and applications in the future.

14.2 Common graphs

A number of families of graphs are so common that they have special names that are worth remembering:

Cliques. A graph on n vertices where every pair of vertices is connected is called a *clique* (or n-clique) and is denoted by K_n. Formally, $K_n = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, j\} : 1 \leq i < j \leq n\}$. The number of edges in K_n is $\binom{n}{2}$.

Paths. A *path* on n vertices, denoted by P_n, is the graph $P_n = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. The number of edges in P_n is $n-1$. The vertices 1 and n are called the *endpoints* of P_n.

Cycles. A *cycle* on $n \geq 3$ vertices is the graph $C_n = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$. The number of edges in C_n is n.

14.3 Some important concepts

Graph isomorphism. If the above definition of a cycle is followed to the letter, a graph is a cycle only if its vertices are natural numbers. So, for example, the graph $G = (V, E)$ with $V = \{A, B, C\}$ and $E = \{\{A, B\}, \{B, C\}, \{C, A\}\}$ would not be a cycle. This seems wrong, because G "looks like" a cycle, and for all practical purposes it is exactly like C_3. The concept of *graph isomorphism* provides a way to formally say that C_3 and G are "the same."

Definition 14.3.1. *Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be* isomorphic *if there exists a bijection $f : V \to V'$ such that*

$$\{x, y\} \in E \text{ if and only if } \{f(x), f(y)\} \in E'.$$

In this case we write $G \equiv G'$ and the function f is called an isomorphism *of G and G'.*

We generally regard isomorphic graphs to be essentially the same, and sometimes do not even draw the distinction. Hence graphs that are isomorphic to cliques, cycles and paths are themselves said to be cliques, cycles and paths, respectively.

Size. The number of edges of a graph is called it *size*. The size of an n-vertex graph is at most $\binom{n}{2}$, achieved by the n-clique.

Degree. The *degree* (or *valency*) of a vertex v in a graph $G = (V, E)$, denoted by $d_G(v)$, is the number of neighbors of v in G. More formally, this degree is

$$d_G(v) = \big|\{u \in V : \{v, u\} \in E\}\big|.$$

A graph in which every vertex has degree k is called *k-regular* and a graph is said to be *regular* if it is k-regular for some k.

The following is sometimes called the Handshake lemma. It can be interpreted as saying that the number of people at a cocktail party who shake hands with an odd number of others is even.

Proposition 14.3.2. *The number of odd-degree vertices in a graph is even.*

Proof. For a graph $G = (V, E)$, consider the sum of the degrees of its vertices:

$$s = \sum_{v \in V} d_G(v).$$

Observe that this sum counts every edge e twice, once for each of the vertices incident to e. Thus $s = 2|E|$, and, in particular, s is even. Subtracting from s the degrees of even-degree vertices of G, we see that the resulting quantity is the sum of the degrees of odd-degree vertices and is still even. This implies the proposition. $\square$

Subgraphs and Connectivity.

Definition 14.3.3. *Given a graph $G = (V, E)$,*

- *A graph $G' = (V', E')$ is said to be a* subgraph *of G if and only if $V' \subseteq V$ and $E' \subseteq E$.*
- *A graph $G' = (V', E')$ is said to be an* induced subgraph *of G if and only if $V' \subseteq V$ and $E' = \{\{u, v\} \in E : u, v \in V'\}$.*

Given a graph G, a path, cycle, or clique in G is a subgraph of G that is a path, cycle, or clique, respectively. Two vertices v and u of G are said to be *connected* if and only if there is a path in G with endpoints u and v. A graph G as a whole is said to be connected if and only if every pair of vertices in G is connected.

A subgraph G' of G is called a *connected component* of G if it is connected and no other graph G'', such that $G' \subset G'' \subseteq G$, is connected. Clearly, a graph is connected if and only if it has a single connected component.

Finally, there is a related notion to a path that is also useful: Given a graph $G = (V, E)$, a *walk* W in G is a sequence $W = (v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n)$ of vertices and edges in G that are not necessarily distinct, such that $\{v_1, v_2, \ldots, v_n\} \subseteq V$, $\{e_1, e_2, \ldots, e_{n-1}\} \subseteq E$, and $e_i = \{v_i, v_{i+1}\}$ for all $1 \leq i \leq n-1$. A walk differs from a path in that vertices and edges can be repeated. The set of edges $\{e_1, e_2, \ldots, e_{n-1}\}$ covered by W is denoted by $E(W)$. Similarly, the set of vertices covered by W is $V(W) = \{v_1, v_2, \ldots, v_n\}$.

14.4 Kinds of graphs

What we have been calling graph is actually only one of many kinds of graphs, namely an undirected, unweighted, simple graph. Let's see how each of these qualities can differ and what other kinds of graphs there are.

A *directed* (simple, unweighted) graph G is an ordered pair (V, E), where V is a set and E is a set of ordered pairs from V. That is,

$$E \subseteq \big\{(x, y) : x, y \in V, x \neq y\big\}.$$

Directed graphs are suitable for modeling one-way streets, non-reflexive relations, hyperlinks in the World Wide Web, and so on. The notion of degree as defined above is no longer applicable to a directed graph. Instead, we speak of the *indegree* and the *outdegree* of a vertex v in G, defined as $|\{u \in V : (u, v) \in E\}|$ and $|\{u \in V : (v, u) \in E\}|$, respectively.

A graph that is not simple can have *multi-edges* and *self-loops*. Multi-edges are multiple edges between the same pair of vertices. (Their presence means that the collection of edges is no longer a set, but a so-called *multiset*.) A self-loop is an edge to and from a single vertex v.

Finally, a graph can also be *weighted*, in the sense that numerical weights are associated with edges. Such weights are extremely useful for modeling distances in transportation networks, congestion in computer networks, etc. We will not dwell on weighted graphs in this course. In fact, unless specified otherwise, the word "graph" will continue to refer to undirected, unweighted, simple graphs.